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Blowing up behavior for a class of nonlinear VIEs connected with parabolic PDEs

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ABSTRACT

We study the blowing-up behavior of solutions of a class of nonlinear integral equations of Volterra type that is connected with parabolic partial differential equations with concentrated nonlinearities. We present some analytic results and, in the case of the kernel of Abel-kind with power nonlinearity and fixed initial data, we give a numerical approximation by using one-point collocation methods.

By means of the numerical simulations, we give the dependence of the blow-up time from the parameters of the equation.

1. Introduction

We want to study a class of nonlinear integral equations of Volterra type (VIEs) related to parabolic partial differential equations (PDEs) with concentrated nonlinearities. We will focus on equations of the type:

$$u(t) = \int_0^t k(t-s)r(s)g(u(s)+h(s))\,\mathrm{d}s \quad t \in [0,T] \tag{1.1}$$

where k is the kernel, r the forcing term, g is the nonlinear source and h the initial data. We will refer to the functions k, r, g, h as the data of the problem and we will assume that:

$$\text{(A1)} \quad \begin{array}{l} k(t-s) \text{ is defined and positive } \forall\, t, s \text{ in the triangle } 0 \le s < t \le T, \\ k(x) \in C^1(]0,T[) \text{ and } k' < 0; \end{array} \tag{1.2}$$

$$\text{(A2)} \quad \begin{array}{l} r(t), h(t) \in C^1([0,T]) \text{ such that} \\ \qquad r(t) \ge 0 \qquad r'(t) \ge 0, \\ \qquad h(t) \ge 0 \qquad h'(t) \ge 0; \end{array} \tag{1.3}$$

$$\text{(A3)} \quad \begin{array}{l} g(x) \in C^1([0,\infty]) \text{ such that} \\ \qquad g(x) > 0 \quad \forall x > 0, \\ \qquad g'(x) > 0 \quad \forall x > 0. \end{array} \tag{1.4}$$

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These equations are widely employed in many fields of applications, such as combustion theory and shear bands formation in steels, see [4,10] and references therein. Other applications are related to the integral representation of the solutions of parabolic PDEs in one dimension, see [23]. There are many interesting issues and open questions concerning the solution of these equations. In particular, the study of the asymptotic behavior (see [17] for an introduction) has been widely investigated in recent years, see the recent review [24,1,18–20,23]. These papers are especially concerned with some characteristics that are very common in the nonlinear analysis such as bifurcation of solutions and asymptotic growth. In particular we will focus our attention on non-existence of the solution after a finite time, known as blow-up, and we report some of the analytical results described in [19,20,25].

Actually, the problem of calculating the exact blow-up time for nonlinear equations is still open and only the case of ordinary differential equations (ODEs) has been successfully solved. This result has been very important for the numerical solution because the knowledge of the blow-up time has allowed the construction of "ad hoc" numerical methods that automatically manage the stopping time, see for example [11]. When the blow-up time is not known, classical numerical methods can fail and there are studies focusing on adaptive methods, see [2,3,5] and references therein. The problem of the detection of blow-up has been also analyzed in the framework of systems of ODEs connected with the semidiscretization in the space of partial differential equations, see [14]. Notice that in the case of the ODEs it is possible to introduce some conditions on the time discretization that guarantees convergence of the numerical blow-up to the exact blow-up time, see [27]. For integral equations, to our knowledge, this subject of the numerical treatment of the blow-up has not been explored in the literature, compare with notes in [7] Section 6.6. For this reason our aim is to investigate the blowing up of VIEs and the behavior of numerical methods when simulating them. We will apply the Niemytzki operator and one point collocation methods for weakly singular Volterra integral equations, as in [6].

By means of the simulations made with these methods we verify a dependence between the parameters and the calculated blow-up time. In the Appendix we report some useful comparison lemmas.

The paper is organized as follows. In Section 2 we introduce the ideas underlying the blow-up behavior of evolution equations by means of the analysis of the ordinary differential equations exhibiting this feature and we summarize the known results for nonlinear Volterra integral equations, focusing on the case of Abel kernel and power nonlinearity. In Section 3 we present our main result and in Section 4 the numerical method utilized and the simulations made. Finally, after some acknowledgments, we conclude with an Appendix containing some comparison lemmas.

2. Existence of solutions and blow-up

First of all, we notice that in the trivial case of $k(t-s) \equiv 1$ Eq. (1.1) can be seen as the integral representation of the first order ordinary differential equation (ODE):

$$u'(t) = r(t)g(u(t) + h(t)). \tag{2.1}$$

As mentioned in the introduction, blow-up in ODEs has been completely investigated. Here we report the main results on this subject in order to use it as a starting point for our further analysis.

Theorem 2.1 (*Blow-up Condition for ODE, [9] (p. 241)*). *Let $y(t)$ be the solution of $y'(t) = G(u(t)),\ y(0) = y_0$, with G positive and continuous in $[y_0, +\infty]$. If:*

$$\int_{y_0}^{\infty} \frac{\mathrm{d}s}{G(s)} < \infty$$

we have that

$$\exists \hat{t} < \infty \quad \text{such that } \lim_{t \to \hat{t}} y(t) = \infty.$$

We will call $\hat{t}$ the ***blow-up time*** *for the solution u.*

This theorem gives the necessary and sufficient condition for blow-up in the ODE theory and the proof gives a technique for deriving the exact blow-up time. Simply, if we can calculate the quantities seen in the previous statement we can write the blow-up time for an ODE:

$$\left.\begin{array}{l} y(0) = y_0 \\ \displaystyle\int_{y_0}^{\infty} \frac{\mathrm{d}s}{G(s)} = K \end{array}\right\} \Rightarrow \text{the blow-up time } \hat{t} \equiv K.$$

Furthermore, for Eq. (2.1) with $h(t) \equiv K_1 \neq 0$ we have:

$$\int_0^t \frac{u'(\tau)}{g(u(\tau) + K_1)} \, \mathrm{d}\tau = \int_0^t r(\tau) \, \mathrm{d}\tau.$$

Now, if f_r is a primitive for the function $r(t)$ and if $\int_{K_1}^{\infty} \frac{\mathrm{d}s}{g(s)} = K_2 < \infty$, then the blow-up time is $\hat{t} \equiv f_r^{-1}(K_2 + f_r(0))$.

While for ODEs the conditions for blow-up and the exact blow-up time are almost trivial, for other equations they are not so simple to obtain. Recently, research has focused on parabolic PDEs, but even in this case there is no relevant result (see Remark 2.3). As we already mentioned in the introduction, the theory on blow-up for VIEs is not very developed and, for this reason, we will focus on the analysis of this problem. First of all we report on the known results for the necessary and sufficient condition for blow-up. Theorem 2.2 (from Ref. [19]) gives this result in the case of kernels of Abel type (i.e. $k(t-s) \equiv (t-s)^{-\alpha}$ with $\alpha \in (0,1)$).

Theorem 2.2. *Let $u(t)$ be a continuous solution of (1.1) where the kernel is of Abel type, and let assumptions* A1–A3 *hold true with the following: $g(0)=0$, $h(0)=0$ and $h(t)>0$ or $h(t)=0\ \forall t>0$, then the maximal solution of problem (1.1) blows up if and only if there exists $\delta>0$ such that:*

$$\int_\delta^\infty \left[\frac{s}{g(s)}\right]^{(1/\alpha)} \frac{\mathrm{d}s}{s} < \infty. \tag{2.2}$$

Remark 2.3 (*Connections with Parabolic PDEs*). In many applications for PDEs on the real line it can be convenient to use the integral representation of the solution. This equation is of the Volterra type with Abel kernel and $\alpha = 1/2$. It is an application of Green's equations, for details see classical literature on parabolic PDEs, for example [15,22].

For a discussion on the connection between nonlinear integral equations and parabolic PDEs with nonlinear flux or concentrated nonlinearities we refer to the survey [23]. For an introduction on blow-up for these PDEs we refer to [22], while the state of the art in such a framework can be found in [3,12,16].

Now we report two results on upper and lower bounds for the blow-up time for a VIE. The proof of the statements below can be found in Ref. [25].

Theorem 2.4 (*Upper Bound for Blow-up Time*). *Let the data functions of Eq. (1.1) be such that assumptions* A1–A3 *are satisfied and, moreover, $g''(x)>0\ \forall x>0$, $h(t)\in[h_0,h_\infty]\ \forall\, 0\le t\le T$ where $0<h_0\le h_\infty<\infty$, then there exist a unique solution $u(t)$ which is increasing.*

Moreover, define $I(t) \equiv \int_{t_0}^t k(t-s)r(s)\,\mathrm{d}s$. If $\exists t^{**}$ such that

$$I(t^{**}) = \int_{h_0}^\infty \frac{\mathrm{d}z}{g(z)} < +\infty, \tag{2.3}$$

then $u(t)$ cannot exist after t^{**}.

Remark 2.5. The previous theorem gives a sufficient condition for the blow-up phenomenon to occur. In fact, if there exists a finite time t^{**} which satisfies (2.3) the equation necessarily explodes before t^{**}.

In the case of data fulfilling both the assumptions of Theorems 2.2 and 2.4, it is easy to see that the two blow-up conditions are equivalent.

Theorem 2.6 (*Lower Bound for Blow-up Time*). *Assume that g and h satisfy the hypothesis of Theorem 2.4, then there exist a continuous function u which is the solution of Eq. (1.1) and such that $u(t)\le M<\infty\ \forall\, t_0\le t<t^*$ with $M<M^*$ where M^* is the smallest solution of:*

$$\frac{M^*}{g(M^*+h_\infty)} = \frac{1}{g'(M^*+h_\infty)}. \tag{2.4}$$

Remark 2.7. Looking at the previous theorem, we can notice that the result is of global existence if the function $I(t)$, which can be seen that is strictly increasing in the hypothesis made in Theorem 2.4, is such that $I(t) < \frac{M^*}{g(M^*+h_\infty)}$. Moreover, we notice that the solution cannot have blow-up if $x\cdot g(x+h_\infty) \ne \frac{1}{g'(x+h_\infty)}\ \forall x>0$.

Remark 2.8 (*Uniqueness*). In the result presented in [25,23] one of the restrictions is that $h(t)>0$. The case $h(t)\equiv 0$ leads to an equation that has non uniqueness of solutions but, in some cases, existence of a non null maximal solution. In the special case of kernel of Abel type it has been proved (see Theorem 2.3 in [19]) that if there exists $\delta>0$ such that:

$$\int_0^\delta \left[\frac{s}{g(s)}\right]^{(1/\alpha)} \frac{\mathrm{d}s}{s} < \infty \tag{2.5}$$

then Eq. (1.1) has a non trivial solution $u^*(t)$ such that $u^*(t)>0\ \forall\, t>0$ and all the other solutions are of the type:

$$u_c(t) = \begin{cases} 0 & \text{if } t<c \\ u^*(t-c) & \text{if } t\ge c. \end{cases}$$

Condition (2.5) for existence of nontrivial solutions is known as Osgood–Gripenberg (O–G), see Eq. 3 in [1] and Refs. [18,20,19].

In the sequel we will consider the particular case of Abel-type kernels with power nonlinearity. We summarize the previous results for these choices of the data in the following remark.

Remark 2.9 (*Abel–Kind with Power Nonlinearity*). Consider the following Volterra–Abel nonlinear equation:

$$u(t) = \int_0^t \frac{(u(s)+\gamma)^p}{(t-s)^\alpha}\,\mathrm{d}s, \quad p > 1, \gamma > 1.$$

Then $u(t)$ blows-up in finite time. Moreover, we can compute in function of p, γ and α the upper bound t^{**} and lower bound t^* for the blow-up time $\hat{t}$ applying Theorems 2.4 and 2.6.

$$\begin{aligned} t^* &= \left[(1-\alpha)\frac{(p-1)^{p-1}}{p^p}\left(\frac{1}{\gamma}\right)^{p-1}\right]^{\frac{1}{1-\alpha}} \\ t^{**} &= \left[(1-\alpha)\frac{1}{p-1}\left(\frac{1}{\gamma}\right)^{p-1}\right]^{\frac{1}{1-\alpha}}. \end{aligned} \tag{2.6}$$

Notice that the estimate t^{**} coincides with the exact blow-up time in the degenerate case of the ODE ($\alpha = 0$). Moreover, the two values differ by a multiplicative factor depending only by p and α:

$$t^* = t^{**}\left[\left(\frac{p-1}{p}\right)^p\right]^{\frac{1}{1-\alpha}}.$$

For these equations the asymptotic behavior is known. Calling $\hat{t}$ the blow-up time, then we have that (see [26]):

$$u(t) \sim K_{\alpha,p}(\hat{t}-t)^{-(1-\alpha)/(p-1)}.$$

3. Main result

Our main result is stated in the following:

Proposition 3.1. *Let $u(t)$ be the solution to:*

$$u(t) = \int_0^t \frac{(u(s)+\gamma)^p}{(t-s)^\alpha}\,\mathrm{d}s, \quad p > 1, \gamma > 1.$$

Then $u(t)$ blows up in finite time $\hat{t}$ such that:

$$\hat{t} = \left[\left(\frac{p-1}{p}\right)^p\right]^{w(\alpha)}\left[(1-\alpha)\frac{1}{p-1}\left(\frac{1}{\gamma}\right)^{p-1}\right]^{\frac{1}{1-\alpha}} \tag{3.1}$$

where $0 \le w(\alpha) \le \frac{1}{1-\alpha}$.

In particular, as seen in the previous section, in [25] are proved upper and lower bounds for the blow-up time that coincide, in the hypothesis of this proposition, with Eq. (3.1) with, respectively $w(\alpha) = 0$ and $w(\alpha) = 1/(1-\alpha)$, see Eq. (2.6). Our idea is to verify in this work, by means of numerical simulation, whether the blow-up time $\hat{t}$ has the behavior of Eq. (3.1) with an exponent w depending only on α.

4. Numerical simulations

As in Remark 2.9, we consider equation:

$$u(t) = \int_0^t \frac{(u(s)+\gamma)^p}{(t-s)^\alpha}\,\mathrm{d}s \tag{4.1}$$

and we solve it by means of the application of the Niemytzki (or substitution) operator. Following [7] (Sections 2.1.5 and 6.2.9) we consider $(\mathbb{N}\phi)(t) \equiv g(t,\phi(t))$. In this way the nonlinear equation can be seen as a Volterra operator $\mathbb{V}$ applied to the substitution of the unknown: $u(t) = (\mathbb{V}\mathbb{N}u)(t)$. Applying the operator $\mathbb{N}$ to both the sides and calling $z(t) = (\mathbb{N}u)(t)$ we obtain an implicitly linear integral equation for $z(t)$: $z(t) = g\left((\mathbb{V}z)(t)\right)$, see also [6]. Summarizing:

$$z(t) = [u(t)+\gamma]^p$$

$$z(t) = \left[\int_0^t \frac{z(s)}{(t-s)^\alpha}\,\mathrm{d}s + \gamma\right]^p. \tag{4.2}$$

Notice that, due to the continuity of the first equation, solution $u(t)$ blows-up in $\hat{t}$ iff $z(t)$ blows-up at the same time $\hat{t}$.

We apply a one-point collocation method to Eq. (4.2), as in [8] or [7] Section 6.2. Our aim is to study the numerical solution of this equation letting α, γ and p be variable. To do this we will carry our simulations with different choices of the discretization and collocation parameter (indicated, respectively with h and θ). We will consider as numerical blow-up time value $\hat{t}^N$ that is the mid point between the last computation carried out and the point where our architecture gives an overflow result. In particular, our simulations are carried out on a Mathematica® workstation that allows one to work up to $NMAX \approx 10^{646456940}$. Knowing the value $\hat{t}^N \equiv \hat{t}^N(\alpha, p, \gamma)$ we calculate:

$$\lambda(\alpha, p, \gamma) = \frac{\log\left(\hat{t}^N(\alpha, p, \gamma)\right) - \log\left(\left[(1-\alpha)\frac{1}{p-1}\left(\frac{1}{\gamma}\right)^{p-1}\right]^{\frac{1}{1-\alpha}}\right)}{\log\left(\left(\frac{p-1}{p}\right)^p\right)}. \tag{4.3}$$

To verify Proposition 3.1, we will see that $\lambda \equiv \lambda(\alpha)$, and so coincides with the exponent $w(\alpha)$ seen in Eq. (3.1).

Let us introduce a grid $t_i,\ i = 1 \dots n,\ t_1 = 0$ and its discretization parameters $h_i = t_{i+1} - t_i$. The basic idea of a one-point collocation method is to search for a piecewise constant solution z^h where the values z_i are such that:

$$z^h(t) = z_i \quad \forall t \in [t_i, t_{i+1}] \quad i \geq 1; \qquad z_i = z^h(t_i + \theta h_i).$$

Using this equation we obtain:

$$\begin{aligned} z_i &= \left[\int_0^{t_i+\theta h_i} \frac{z^h(s)}{(t_i+\theta h_i - s)^\alpha}\,\mathrm{d}s + \gamma\right]^p \\ &= \left[\int_{t_i}^{t_i+\theta h_i} \frac{z^h(s)}{(t_i+\theta h_i - s)^\alpha}\,\mathrm{d}s + \int_{t_0}^{t_i} \frac{z^h(s)}{(t_i+\theta h_i - s)^\alpha}\,\mathrm{d}s + \gamma\right]^p \\ &= \left[\int_0^{\theta h_i} \frac{z_i}{(\theta h_i - \sigma)^\alpha}\,\mathrm{d}\sigma + \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{z_j}{(t_i+\theta h_i - s)^\alpha}\,\mathrm{d}s + \gamma\right]^p. \end{aligned}$$

Now, consider:

$$\begin{aligned} &A(\theta, \alpha, h_i) = \frac{(\theta h_i)^{1-\alpha}}{1-\alpha} \\ &L(\theta, \alpha, \gamma, z_0, \dots, z_{i-1}, h_i, t_0, \dots, t_i) = \gamma + \sum_{j=0}^{i-1} z_j \frac{\left(t_i+\theta h_i - t_j\right)^{1-\alpha} - \left(t_i + \theta h_i - t_{j+1}\right)^{1-\alpha}}{1-\alpha}. \end{aligned} \tag{4.4}$$

We have that the solution z_i can be written in the form $z_i = [Az_i + L]^p$. To solve this nonlinear equation by means of a fixed point iteration process ($z_i^{(j+1)} = [Az_i^{(j)} + L]^p$), we need to ensure a contraction property,[1] as done, for example, in [2]. This is made in our simulations by means of the introduction of a relative tolerance *tol*. The main features of our algorithm are summarized in the following:

Initializations : Fix h_i and compute $t_i\ \forall i = 1, \dots, n$
First Integration Step : $z_1^{(0)} = \gamma;\ k = 0;$ compute A from Eq. (4.4)
Fixed Point Iterations : **while** $(|z_1^{k-1} - z_1^k| > tol \cdot |z_1^k|$ and $k \leq k_{\max})$
  $z_1^{(k+1)} = \left(Az_1^{(k)} + \gamma\right)^p;\ k = k+1$
 end while
Next Integration Steps : **for** $i = 2, \dots, n$
Initializations and computations : $z_i^{(0)} = z_{i-1};\ k = 0;$ compute A and L from Eq. (4.4)
Fixed Point Iterations : **while** $(|z_i^{k-1} - z_i^k| > tol \cdot |z_i^k|$ and $k \leq k_{\max})$
  $z_i^{(k+1)} = \left(Az_i^{(k)} + L\right)^p;\ k = k+1$
 end while
Control on blow-up : **if** $(z_i^{(k)} = Overflow)$, $t^N = (t_i - t_{i-1})/2$, **break**
  else $z_i = z_i^{(k)}$
 end if
end for

[Output: Numerical solution z_i for Eq. (4.2);
Calculated blow-up time t^N.]

[1] Notice that a sufficient condition for convergence is:

$$z_i^{(j)} < \frac{1}{A}\left[\left(\frac{1}{Ap}\right)^{\frac{1}{p-1}} - L\right].$$

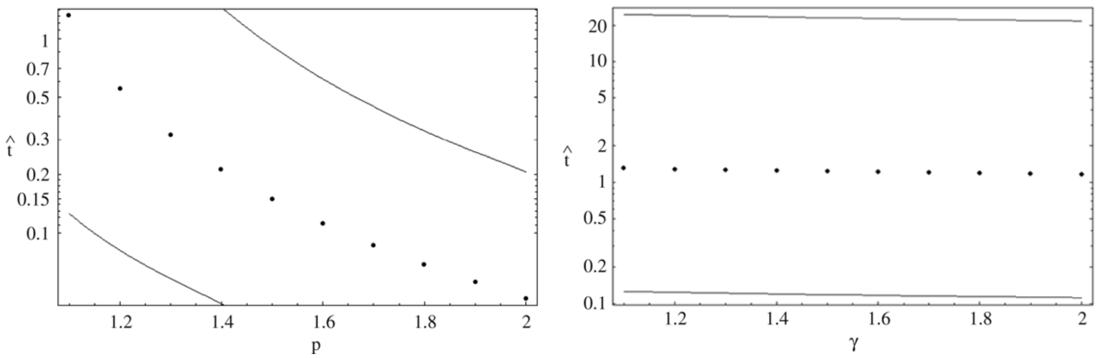

Fig. 1. Blow-up times (dot lines) and estimates (solid lines) in log scale. On the left the case of $\gamma = 1.1$, $\alpha = 0.5$ and $p \in [1.1, 2]$; on the right $p = 1.1$, $\alpha = 0.5$ and $\gamma \in [1.1, 2]$.

Table 1
Calculated blow-up times

p	γ 1.1	1.2	1.3	1.4	1.5
Case 1					
1.1	8.1315	8.0535	7.9825	7.9175	7.8565
1.2	3.9665	3.8905	3.8215	3.7585	3.7015
1.3	2.5835	2.5095	2.4435	2.3835	2.3295
1.4	1.8955	1.8235	1.7595	1.7025	1.6505
1.5	1.4845	1.4145	1.3525	1.2975	1.2495
Case 2					
1.1	1.3085	1.2855	1.2655	1.2465	1.2295
1.2	0.5535	0.5345	0.5175	0.5025	0.4885
1.3	0.3185	0.3015	0.2875	0.2745	0.2635
1.4	0.2085	0.1945	0.1825	0.1715	0.1625
1.5	0.1475	0.1355	0.1245	0.1155	0.1075
Case 3					
1.1	0.047515	0.046155	0.044935	0.043845	0.042845
1.2	0.015805	0.014915	0.014135	0.013455	0.012845
1.3	0.007415	0.006795	0.006265	0.005815	0.005425
1.4	0.004075	0.003625	0.003255	0.002945	0.002685
1.5	0.002465	0.002125	0.001855	0.001635	0.001455

Case 1. $\alpha = 0.1$ where $\lambda \approx 0.125$; Case 2. $\alpha = 0.5$ where $\lambda \approx 1.09$; Case 3. $\alpha = 0.7$ where $\lambda \approx 2.52$.

In our simulations we always consider $tol = 10^{-3}$ and $k_{\max} = 40$. If the convergence of the fixed point iterations fails, i.e. the stepsize is too large, we continue the computation with a new fixed stepsize that is obtained by reducing the previous value by a factor 10^{-1}. With this strategy the fixed point iterations always converge before $k_{\max}$.

In Fig. 1 we plot the calculated blow-up times and the upper and lower estimates (2.6) with fixed α, γ and various p and fixed α, p and various γ.

In these simulations $h_i \equiv h$ is fixed to the value 10^{-3} and the collocation parameter considered is $\theta = 0.5$. We have applied a fixed spacing that is not well suited for blow-up problems, compare with [27], but it is not yet clear what would be an appropriate strategy for the automatic (or even a priori) computation of a variable step-size, compare with [7] notes in Section 6.6 (see also below our comment on graded mesh).

In Table 1 we report the calculated blow-up times with $\alpha = 0.1, 0.5, 0.7$. Corresponding to these simulations we calculate λ as in Eq. (4.3). This value is almost constant for each table, in particular the variations are in the third significant digit.[2]

These simulations are carried out considering in the algorithm $\theta = 0.5$, $tol = 10^{-4}$ and $h = 10^{-3}$ in the first two cases while $h = 10^{-5}$ in the last.

We have also considered other choices of the parameters of the method (h, θ, tol), but the results can be considered the same for our point of view.

We can notice, although, that as θ is changed the calculated solution varies with a monotonic law, see Fig. 2. This seems to be related to the monotonicity of the power nonlinearity, see the Appendix. Further analysis of the properties of the method will be the object of future work.

[2] Note that a small variation on the λ produces a small variation on the blow-up time.

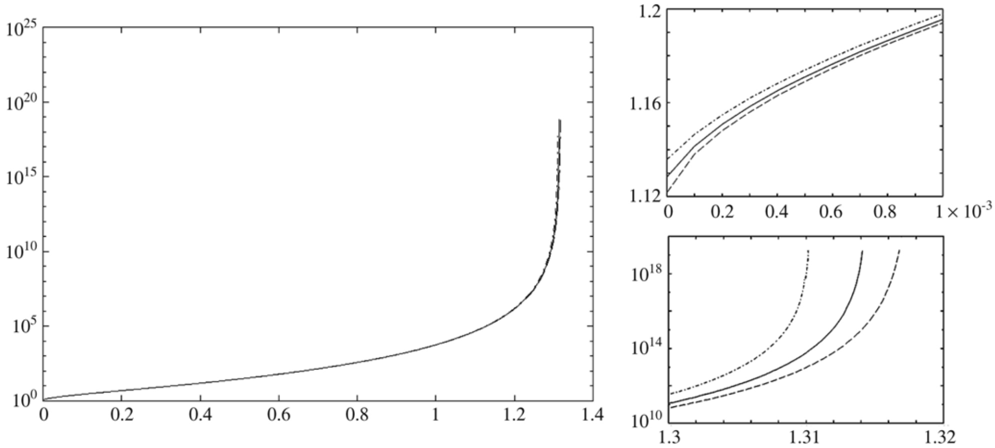

Fig. 2. Numerical solution with $\theta = 0.1$ (dash–dot line), $\theta = 0.5$ (solid line) and $\theta = 1$ (dashed line). On the left the solution everywhere in log scale. On the right two expanded views: near $t = 0$ (up) in linear scale and near $t = \hat{t}^N$ (bottom) in log scale.

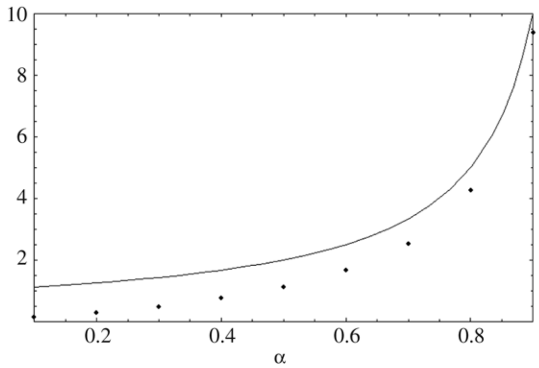

Fig. 3. Calculated values of function $w(\alpha)$ (dot line) and its upper bound (solid line).

Also the case of finer or graded meshes has been considered. Finer meshes give almost the same result and computations become, in some cases, very heavy.

Graded meshes are usually employed in the case of weakly singular equations and lead to an efficient algorithm that considers a variable h_i that is finer near the first point, see [7] Section 6.2.3. In our case we need a finer computation near the blow-up point, and for this reason we calculate first (with an equispaced mesh) an estimate $\tilde{t}$ of the blow-up time and then take:

$$t_i = \tilde{t}\left[1 - \left(\frac{n-i}{n}\right)^{\frac{1}{1-\alpha}}\right].$$

The results are similar to the ones obtained with fine meshes and seem to confirm improved behavior of the method. This argument will be covered in a subsequent work.

In the last plot (Fig. 3) we consider the values of function $\lambda(\alpha, 1.1, 1.1) \equiv w(\alpha)$. These values are obtained with $\theta = 0.5$. From these simulations we are not able to give any conjecture on the exact rule of the function $w(\alpha)$.

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Appendix. Some comparison lemmas

In this Appendix we present two comparison lemmas involving the data of Eq. (1.1) with the assumptions A1–A3 and a result that applies in the case of Eq. (4.1). We will also assume that g and h satisfy the hypothesis for the uniqueness of the solution (see Remark 2.8 and Theorem 2.2). We omit the proofs of these results that can be easily derived form the analysis in [13,17,21].

Lemma A.1 (*Comparison 1, with the Source*). *Let $u_1(t)$ and $u_2(t)$ be solutions of:*

$$u_1(t) = \int_0^t k(t-s)r(s)g_1(u(s)+h(s))\,\mathrm{d}s$$
$$u_2(t) = \int_0^t k(t-s)r(s)g_2(v(s)+h(s))\,\mathrm{d}s$$

with

$$g_1(\xi) \le g_2(\xi) \quad \forall \xi > 0$$
$$g'(\xi) \le g_2'(\xi) \quad \forall \xi > 0$$
$$g_1', g_2' \text{ increasing},$$

then $u_1(t) < u_2(t)\ \forall t > 0$.

Lemma A.2 (*Comparison 2, with the Initial Data*). *Let $u_1(t)$ and $u_2(t)$ be solutions of:*

$$u_1(t) = \int_0^t k(t-s)r(s)g(u(s)+h_1(s))\,\mathrm{d}s$$
$$u_2(t) = \int_0^t k(t-s)r(s)g(v(s)+h_2(s))\,\mathrm{d}s$$

with

$$h_1(0) < h_2(0)$$
$$h_1(\tau) \le h_2(\tau) \quad \forall \tau > 0$$
$$h_1'(\tau) \le h_2'(\tau) \quad \forall \tau > 0$$

In this hypothesis $u_1(t) < u_2(t)\ \forall t > 0$.

Lemma A.3 (*Comparison 3*). *Consider:*

$$u_1(t) = \int_0^t \frac{(u_1(s)+\gamma_1)^p}{(t-s)^\alpha}\,\mathrm{d}s$$
$$u_2(t) = \int_0^t C_2\frac{(u_2(s)+\gamma_2)^p}{(t-s)^\alpha}\,\mathrm{d}s.$$

Then u_1 and u_2 blow-up in the same $\hat{t}$ iff

$$\gamma_1 = \gamma_2 C_2^{\frac{1}{p-1}}.$$

Proof. Simply manipulating the equation solved by u_1 we obtain:

$$u_1(t) = \int_0^t \frac{(u_1(s)+\gamma_1)}{(t-s)^\alpha}\,\mathrm{d}s \Rightarrow$$
$$Ku_1(t) = \int_0^t \frac{(Ku_1(s)+\gamma_1)^p}{(t-s)^\alpha}\,\mathrm{d}s \Rightarrow$$
$$u_1(t) = \int_0^t K^{p-1}\frac{(u_1(s)+\gamma_1 K^{-1})^p}{(t-s)^\alpha}\,\mathrm{d}s$$

now, call $C_2 = K^{p-1}$, we have that:

$$u_1(t) = \int_0^t C_2\frac{\left(u_1(s)+\gamma_1 C_2^{-\frac{1}{p-1}}\right)^p}{(t-s)^\alpha}\,\mathrm{d}s$$

and this is exactly the same equation solved by u_2 with the choice of $\gamma_1 = \gamma_2 C_2^{\frac{1}{p-1}}$. $\square$

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